FROM FIXED-ENERGY MSA TO DYNAMICAL LOCALIZATION: A CONTINUING QUEST FOR ELEMENTARY PROOFS

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Abstract. We review several techniques and ideas initiated by a remarkable work by Spencer [26], used and further developed in numerous subsequent researches. We also describe a relatively short and elementary derivation of the spectral and strong dynamical Anderson localization from the fixed-energy analysis of the Green functions, obtained either by the Multi-Scale Analysis (MSA) or by the Fractional-Moment Method (FMM). This derivation goes in the same direction as the Simon–Wolf criterion [28], but provides quantitative estimates, applies also to multi-particle models and, combined with a simplified variant of the Germinet–Klein argument [20], results in an elementary proof of dynamical localization.

1. Introduction.

The mathematical theory of Anderson localization, describing the motion a quantum particle (or a collection of non-interacting particles) in a disordered environment has reached by now its age of maturity. The number of different mathematical models (different even from the point of view of applications to physical systems) and technical tools, allowing to analyze these models, is quite impressive. On the other hand, this also makes mastering these techniques difficult for the beginners. It is not always easy to see the main ideas behind dozens of pages filled with definitions, preliminary facts and sophisticated arguments pushed to the extreme due to the complexity of the problem. Yet, the maturity of a mathematical theory can also be judged by the presence of comprehensive techniques and simple general principles, guiding one through the jungle of more complex models and methods, so a number of leading researchers in this area of mathematical physics have been conducting a quest for simpler, more elementary proofs of localization, intuitive (yet rigorous) techniques and principles.

The quest began already in late 1980's, when Simon and Wolf [28] proved that suitable fixed-energy bounds on the Green functions imply a.s. pure point spectrum, and then Spencer proposed in a remarkably short paper [26] an elementary reformulation of the fixed-energy MSA developed in his pioneering joint work with Fröhlich [16]. Besides the fact that the paper [26] came as a perfect complement to the Simon–Wolf argument, it draws the reader's attention to the parallels between the theory of random operators and a more traditional spectral analysis of almost periodic operators, following the classical works on periodic operators.

The first paper on the FMM, published by Aizenman and Molchanov [1] in 1993, a few years after the cycle [12, 16, 17, 26, 28], also has been an important event in the quest for elementary proofs (which the title of [1] makes explicit). It took some time and efforts to complete the first stage of the FMM (a fixed-energy analysis

of fractional moments of the resolvents) with additional arguments leading – in a simple way – to the strong dynamical localization. One can only regret that the MSA and the FMM have been evolving in parallel with a very weak interaction, over the last two decades.

In the present paper, I am going to give a short and certainly incomplete review of techniques and ideas brought to life or initiated indirectly by Tom Spencer in his short article [26] which has the good fortune and privilege to belong to those works which "*tell much more than they say*".

The present text is intentionally left relatively short, to keep up with the spirit of [26] (which was 10 page *short*). For this reason, interesting applications of the MSA to DSO with quasi-periodic (and more generally, deterministic) potentials had to be omitted; these include deep analytic works by Bourgain, Goldstein and Schlag (cf., e.g., [4, 5]), where the MSA techniques were applied to lattice Schrödinger operators with "analytic" potentials, and a recent article [7] where a different MSA-based approach has been used to treat parametric families of almost periodic and some other *deterministic* operators by traditional – and very simple – methods of the theory of *random* operators. On a personal note, I have to acknowledge with pleasure numerous fruitful discussions on this subject with Tom Spencer.

The pioneering work [17] on Anderson localization in multidimensional disordered media (following the work by Fröhlich and Spencer [16]) proved only the exponential *spectral* localization, i.e., pure point spectrum and exponential decay of generalized eigenfunctions. The relations between the spectral and dynamical manifestations of the Anderson localization phenomenon have been studied later in the work [24] which has influenced further development by Germinet–De Bièvre [19] and by Damanik–Stollmann [13]. The overall result of these researches (perfectly summarized by the title of the paper [13]) was a clear understanding that the variable-energy MSA (VEMSA, in short) provides a sufficient input for the proof of the strong dynamical localization, in discrete and continuous random media.

Germinet and Klein [20] made a further step and gave a much shorter derivation of the dynamical localization from the key MSA estimates, avoiding a tedious analysis of the random geometry of the so-called centers of localization (the latter notion essentially goes back to [24]). The elegance of their approach resides, in particular, in the fact that the eigenfunction correlator bounds are inferred from those provided by the MSA directly in the entire configuration space (a Euclidean space, in their case). However, this elegance comes with a price: one has to rely upon a deep analysis of weighted Hilbert–Schmidt norms of spectral projections of Schrödinger (or some other) operators in $L^2(\mathbb{R}^d)$. Such an analysis has been carried out by Simon [25] for potentials bounded from below by $-C(|x|+1)$, and later extended by Poerschke and Stolz [27] to potentials bounded from below by $-C(|x|^2+1)$; the latter is a usual condition for the essential self-adjointness of the respective Schrödinger operator. In arbitrary *finite* volumes, the analog of Germinet–Klein argument is reduced to a three-word instruction: "*Apply Bessel's inequality*".

The situation is particularly simple for operators on a countable graph, where functional analysis is in fact replaced by linear algebra. Indeed, the required eigenfunction expansions are completely elementary here, for Hermitian operators in finite-dimensional Hilbert spaces. In continuous configuration spaces (Euclidean spaces and quantum graphs), a similar effect is achieved whenever the random

operators in question have (as they usually do) compact resolvent in any finite volume, under reasonable requirements on its geometry. Physically speaking, only such finite-volume bounds (uniform in the size of the volume) are relevant for applications to the quantum transport in disordered media. The term "finite volume" should not be misleading: a sample of a random media of the size of the Milky Way is still finite ... and the task of designing computer processors (let alone nano-devices) of *that* size does not occupy yet the minds in the physics community.

Mathematically speaking, once uniform bounds on eigenfunction correlators are obtained in finite volumes of arbitrarily large size, one is just three words away from the strong dynamical localization in the entire space: "*Apply Fatou lemma*" (cf., e.g., [2], [3]).

With these observations in mind, I propose here a streamlined derivation of the VEMSA-type probabilistic bounds from their simpler FEMSA counterparts. Several elements of such a derivation appeared earlier, e.g., in [23], [14] (but the approach from Section 7 appears to be original). It plays a role similar to that of the Simon–Wolf argument, providing a 'soft way' from the fixed-energy localization to stronger manifestations of the Anderson localization phenomenon. The key notion here is what is called in Sections 6–7 the "singular width" of the spectrum: the total Lebesgue measure of a (reasonably large) finite number of intervals chosen in such a way that outside this "singular zone" Green functions are exponentially small. (Curiously, the abbreviation "SW" suits to "Simon–Wolf", "singular width" and "soft way"; the central symetry transforms it into "MS", as in "Martinelli–Scoppola", cf. [23].)

The structure of this paper is as follows.

- Main notions and notations are introduced in Section 2.
- In Sections 3–4, we give a streamlined version of the analytic core of the fixed-energy MSA, in the simplest form going back to [26], but formulated in a more abstract fashion.
- In Section 5, we show that the simple, fixed-energy analysis from [26] can be easily improved so as to provide the key probabilistic bounds on the Green functions stronger than any power law.
- In Section 6, following essentially[1] [14], we derive from fixed-energy bounds their variable-energy counterparts. The obtained results apply also to the FMM-type bounds (which are always initially obtained at fixed energy).
- In Section 7, we give another derivation of the variable-energy bounds which allows to prove directly the exponential spectral localization (i.e., the exponential decay of eigenfunctions), as well as the dynamical localization, under a stronger assumption on the random potential (cf. Eqn (7.2)).
- In Section 8, we formulate a finite-volume variant of the Germinet–Klein argument, allowing to derive the strong dynamical localization from variable-energy bounds on the resolvents (obtained by the MSA or by the FMM).
- Section 9 describes a simple adaptation of the techniques from Section 5 sufficient for the proof of sub-exponential dynamical localization.
- In Section 10, we briefly describe another simple approach (developed in our recent paper [8]) which also has its roots in [26]. However, the main

[1] I thank Tom Spencer and Sasha Sodin for a fruitful discussion of the works [14]–[15].

object of the scale induction is here the decay of the eigenfunctions in finite balls, rather than the decay of Green functions.

The principal statements are theorems 6.1, 6.2, 8.1, 9.5 and 9.6.

For the sake of brevity and clarity of presentation, I do not discuss several powerful (but more complex) techniques from the works by Germinet–Klein, including the bootstrap MSA (cf. [20]) and spectral reductions from [21] used in a very general framework of random operators in $\mathbb{R}^d$ with singular probability distributions.

To conclude the introduction, I would like to emphasize the role that the paper [26] has played in a recent development of the multi-particle MSA (MPMSA). In our joint works with Yuri Suhov (cf., e.g., [6]), we aimed initially to prove the spectral localization, which requires traditionally a variable-energy analysis. However, the fixed-energy analysis has a substantial advantage to simplify both geometrical and analytical ingredients of the MPMSA. I plan to address this subject in a forthcoming work, using the reductions described in Sections 6–8.

2. Basic notations, facts and assumptions

Throughout this paper, we work with discrete Schrödinger operators (DSO) acting in Hilbert spaces of square-summable complex functions on connected countable graphs. Indeed, the techniques and results of the MSA, initially developed for operators on periodic lattices, are naturally extended to more general graphs with polynomially bounded growth of balls (such graphs as Bethe lattices remain so far out of the MSA's reach). Another motivation for presenting the new approach on a graph comes from the fact that the natural language for the description of a system of $N > 1$ interacting indistinguishable quantum particles (bosons or fermions) is that of a symmetric power of the configuration space $\mathcal{Z}$ of the respective single-particle system; already in the case where the configuration space is $\mathcal{Z} = \mathbb{Z}^d$, $d > 1$, its N-th *symmetric* power is no longer a periodic lattice.

Consider a finite or countable connected graph $(\mathcal{G}, \mathcal{E})$, with the set of vertices $\mathcal{G}$ and the set of edges $\mathcal{E}$; for brevity, we will often call $\mathcal{G}$ the graph, omitting the reference to $\mathcal{E}$. We denote by $\mathrm{d}_{\mathcal{G}}(\cdot, \cdot)$ (sometimes simply by $\mathrm{d}(\cdot, \cdot)$) the canonical distance on the graph $\mathcal{G}$: $\mathrm{d}_{\mathcal{G}}(x, y)$ is the length of the shortest path $x \rightsquigarrow y$ over the edges. We will assume that the growth of balls $\mathrm{B}_L(x) := \{y : \mathrm{d}_{\mathcal{G}}(x, y) \leq L\}$ is polynomially bounded:

$$\sup_{x \in \mathcal{G}} |\mathrm{B}_L(x)| \leq C_d L^d, \quad L \geq 1. \tag{2.1}$$

In particular, the coordination number $n_{\mathcal{G}}(x) := \{y : \mathrm{d}_{\mathcal{G}}(x, y) = 1\}$ of any vertex x is bounded by C_d (even by $C_d - 1$). The canonical (negative) graph Laplacian $(-\Delta_{\mathcal{G}})$ on a finite or countable graph $(\mathcal{G}, \mathcal{E})$ is given by

$$(-\Delta_{\mathcal{G}} f)(x) = \sum_{\langle x, y \rangle} (f(x) - f(y)) = n_{\mathcal{G}}(x) f(x) - \sum_{\langle x, y \rangle} f(y) \tag{2.2}$$

where we use a popular notation $\langle x, y \rangle$ for a pair of nearest neighbors $x, y \in \mathcal{G}$, i.e., $\mathrm{d}_{\mathcal{G}}(x, y) = 1$, and $n_{\mathcal{G}}(x)$ is the coordination number of the point x. For brevity, we will sometimes use slightly abusive notations like $\langle x, y \rangle \in \Lambda$, $\Lambda \subset \mathcal{G}$ instead of $\langle x, y \rangle \in (\Lambda \times \Lambda) \cap \mathcal{E}_{\mathcal{G}}$.

From this point on, unless otherwise specified, we will use the notation $\mathcal{G}$ only for finite connected graphs, while $\mathcal{Z}$ will stand for a countable connected graph with

polynomial growth of balls. In operator form, we can write

$$-\Delta_{\mathcal{G}} = n_{\mathcal{G}} - \sum_{\langle x,y\rangle} \Gamma_{x,y}, \quad \Gamma_{x,y} = |\,\mathbf{1}_x\rangle\langle\mathbf{1}_y\,|,$$

where $n_{\mathcal{G}}$ is the operator of multiplication by the function $x \mapsto n_{\mathcal{G}}(x)$. Given a proper (connected) subgraph $\Lambda \subsetneq \mathcal{G}$, define its internal, external and the so-called edge boundary (relative to $\mathcal{G}$) as follows:

$$\begin{aligned}\partial^-_{\mathcal{G}}\Lambda &= \{y \in \Lambda :\, \mathrm{d}_{\mathcal{G}}(x, \mathcal{G}\setminus\Lambda) = 1\}, \quad \partial^+_{\mathcal{G}}\Lambda = \partial^-_{\mathcal{G}}\mathcal{G}\setminus\Lambda,\\ \partial_{\mathcal{G}}\Lambda &= \{(x,y) \in \partial^-_{\mathcal{G}}\Lambda \times \partial^+_{\mathcal{G}}\Lambda :\, \mathrm{d}_{\mathcal{G}}(x,y) = 1\}.\end{aligned}$$

Working with a given graph $\mathcal{G}(\subset \mathcal{Z})$, we always mean by a ball $\mathrm{B}_R(u) \subset \mathcal{G}$ the set $\{y \in \mathcal{G} :\, \mathrm{d}_{\mathcal{G}}(u,y) \le R\}$, i.e., the **ball relative to the metric space** $(\mathcal{G}, \mathrm{d}_{\mathcal{G}})$.

The Laplacian (hence, a DSO) in a subgraph $\Lambda \subset \mathcal{G}$ can be defined in various ways. The two most popular choices are:

- The canonical (negative) Laplacian in Λ, $(-\Delta^{\mathrm{N}}_{\Lambda} f) = (-\Delta_{\Lambda} f)$, defined as in (2.2) with $\mathcal{G}$ replaced by Λ. It this context, it is usually considered as an analog of the Neumann Laplacian, and reads as follows:

$$(-\Delta^{\mathrm{N}}_{\Lambda} f)(x) = n_{\Lambda}(x) - \sum_{\langle x,y\rangle \in \Lambda} f(y). \tag{2.3}$$

- The Dirichlet Laplacian $(-\Delta^{\mathrm{D}}_{\Lambda,\mathcal{G}}) = \mathbf{1}_{\Lambda}(-\Delta^{\mathrm{D}}_{\Lambda,\mathcal{G}})\,\mathbf{1}_{\Lambda} \restriction \ell^2(\Lambda)$. Here we use a natural injection $\ell^2(\Lambda) \hookrightarrow \ell^2(\mathcal{G})$. The Dirichlet counterpart of (2.4) is

$$(-\Delta^{\mathrm{D}}_{\Lambda} f)(x) = n_{\mathcal{G}}(x) - \sum_{\langle x,y\rangle \in \Lambda} f(y), \tag{2.4}$$

  with $n_{\mathcal{G}}(x) \ge n_{\Lambda}(x)$, so $(-\Delta^{\mathrm{D}}_{\Lambda}) \ge (-\Delta^{\mathrm{N}}_{\Lambda})$ in the sense of quadratic forms.

We will use the Dirichlet Laplacians and DSO H^{D}_{Λ}. Given a decomposition $\mathcal{G} = \Lambda \sqcup \Lambda^{\mathrm{c}}$, $\Lambda^{\mathrm{c}} := \mathcal{G}\setminus\Lambda^{\mathrm{c}}$, we can write

$$\begin{aligned}-\Delta^{\mathrm{D}}_{\mathcal{G}} &= n_{\mathcal{G}} - \sum_{\langle x,y\rangle\in\Lambda} \Gamma_{x,y} - \sum_{\langle x,y\rangle\in\Lambda^{\mathrm{c}}} \Gamma_{x,y} - \sum_{\langle x,y\rangle\in\partial\Lambda} (\Gamma_{x,y} + \Gamma_{y,x})\\ &= \big((-\Delta^{\mathrm{D}}_{\Lambda}) \oplus (-\Delta^{\mathrm{D}}_{\Lambda^{\mathrm{c}}})\big) - \Gamma_{\Lambda,\mathcal{G}}\end{aligned}$$

with $\Gamma_{\Lambda,\mathcal{G}} = \sum_{\langle x,y\rangle\in\partial\Lambda} (\Gamma_{x,y} + \Gamma_{y,x})$. Respectively for the DSO $H_{\mathcal{G}} = -\Delta^{\mathrm{D}}_{\mathcal{G}} + V$, where $V : \mathcal{G} \to \mathbb{R}$ is usually referred to as the potential, one has

$$H_{\mathcal{G}} = H^{\bullet}_{\mathcal{G},\Lambda} - \Gamma_{\Lambda,\mathcal{G}}, \qquad H^{\bullet}_{\mathcal{G},\Lambda} := (-\Delta^{\mathrm{D}}_{\Lambda} + V) \oplus (-\Delta^{\mathrm{D}}_{\Lambda^{\mathrm{c}}} + V).$$

We omit the superscript "N", since the nature of the boundary conditions in $\mathcal{G}$ is not related to the choice of Dirichlet or Neumann decoupling induced by $\mathcal{G} = \Lambda\sqcup\Lambda^{\mathrm{c}}$.

The spectrum of a (finite-dimensional) operator $H_{\mathcal{G}}$, i.e., the set of its eigenvalues (EVs) counted with multiplicities, will be denoted by $\Sigma(H_{\mathcal{G}})$.

In a number of formulae and statements, we will use the parameters $\beta, \tau, \varrho \in (0,1)$, and $\alpha \in (1,2)$. Unless otherwise specified, we assume that $\beta = 1/2$, $\tau = 1/8$, $\varrho = (\alpha-1)/2 = 1/6$ and $\alpha = 3/2$. Note that the exponent $\frac{1+\varrho}{\alpha}$ figuring in Definition 2.1 then equals 7/8. (These settings **will be changed in Section** 9.)

Definition 2.1. Given numbers $E \in \mathbb{R}$ and $m > 0$, a ball $\mathrm{B}_L(u)$ is called

- E-resonant (E-R, in short), if $\mathrm{dist}(\Sigma(H_{\mathrm{B}_L(u)}), E) < \mathrm{e}^{-L^{\beta}}$, and E-nonresonant (E-NR), otherwise;

- (E,m)-nonsingular ((E,m)-NS), if for all $x,y\in \mathrm{B}_L(u)$ with $\mathrm{d}(x,y)\ge L^{\frac{1+\varrho}{\alpha}}$

$$C_d^2 L^d \cdot |G_{\mathrm{B}_L(u)}(x,y;E)| \le \mathrm{e}^{-\gamma(m,L)\mathrm{d}(x,y)}, \tag{2.5}$$

where

$$\gamma(m,L) := m(1+L^{-\tau}), \tag{2.6}$$

and (E,m)-nonsingular ((E,m)-NS), otherwise.

Observe that for any ball $\mathrm{B}_L(u)$, $|\partial \mathrm{B}_L(u)| \le C_d^2 L^d$, by virtue of (2.1).

2.1. **Assumptions on the random potential.** For clarity of presentation, we always assume that the random potential field $V:\mathcal{Z}\times\Omega\to\mathbb{R}$ on a graph $\mathcal{Z}$ is IID, with Lipshitz continuous marginal probability distribution function (PDF) F_V:

$$\sup_{t\in\mathbb{R}}(F_V(t+s)-F_V(t)) \le C_W s, \quad C_W\in(0,+\infty). \tag{2.7}$$

It is well-known that this assumption can often be relaxed to uniform Hölder continuity, and even to a form of log-Hölder continuity.

2.2. **The Wegner estimate.** The original result by Wegner [29] has been adapted to a large number of classes of random operators. Here we apply its simplest version, for DSO with a Lipshitz continuous IID random potential. The proof can be found in a number of books and review articles; cf., e.g., Lemma VIII.1.8 in [10].

Lemma 2.1 (Wegner estimate)**.** *Under the assumption* (2.7)*, for any finite graph* $\mathcal{G}$ *of cardinality* $|\mathcal{G}|$ *and any* $\epsilon\in[0,1]$

$$\sup_{E\in\mathbb{R}} \mathbb{P}\left\{ \operatorname{dist}(\Sigma(H_{\mathcal{G}}),E)\le\epsilon \right\} \le C_W|\mathcal{G}|\epsilon. \tag{2.8}$$

In fact, the above statement remains valid for any ensemble of random operators of the form $V(\,\cdot\,;\omega)+H_0$ with a non-random operator H_0, for only the diagonal part $V:\mathcal{G}\times\Omega\to\mathbb{R}$ is used in the proof (cf. [10]).

3. Decoupling of resolvents on graphs

3.1. **Geometric resolvent inequality.** The second resolvent identity implies the so-called Geometric resolvent equation for the resolvents $G_{\mathcal{G}}(E)=(H_{\mathcal{G}}-E)^{-1}$, $G_{\Lambda^{\mathrm{c}}}(E)=(H_{\Lambda^{\mathrm{c}}}-E)^{-1}$, $G^{\bullet}_{\mathcal{G},\Lambda}(E)=(H^{\bullet}_{\mathcal{G},\Lambda}-E)^{-1}$:

$$G_{\mathcal{G}}(E) = G^{\bullet}_{\mathcal{G}}(E) + G^{\bullet}_{\mathcal{G}}(E)\,\Gamma_{\Lambda,\mathcal{G}}\,G_{\mathcal{G}}(E). \tag{3.1}$$

For $x,u\in\Lambda$ and $y\in\Lambda^{\mathrm{c}}$, one has $G^{\bullet}_{\mathcal{G}}(x,u;E)=G^{\mathrm{D}}_{\Lambda}(x,u;E)$ and $G^{\bullet}_{\mathcal{G}}(x,y;E)=0$. This results in the Geometric resolvent equation for the Green functions

$$G_{\mathcal{G}}(x,y;E) = \sum_{\langle u,u'\rangle\in\partial\mathcal{G}\Lambda} G^{\mathrm{D}}_{\Lambda}(x,u;E)\,G_{\mathcal{G}}(u',y;E) \tag{3.2}$$

and the Geometric resolvent inequality (GRI)

$$|G_{\mathcal{G}}(x,y;E)| \le \sum_{\langle u,u'\rangle\in\partial\mathcal{G}\Lambda} |G^{\mathrm{D}}_{\Lambda}(x,u;E)|\,|G_{\mathcal{G}}(u',y;E)|. \tag{3.3}$$

4. Subharmonicity on graphs

4.1. **Regular subharmonic functions.**

Definition 4.1. Let $\mathcal{G}$ be a finite connected graph, $L \geq \ell \geq 0$ two integers and $q \in (0,1)$. A function $f : \mathcal{G} \to \mathbb{R}_+$ is called (ℓ, q)-subharmonic in a ball $\mathrm{B}_L(u) \subsetneq \mathcal{G}$ if for any ball $\mathrm{B}_\ell(x) \subseteq \mathrm{B}_L(u)$ one has

$$f(x) \leq q \max_{y \in \mathrm{B}_{\ell+1}(x)} f(y). \tag{4.1}$$

We will often use the notation $\mathcal{M}(f, \Lambda) := \max_{x \in \Lambda} |f(x)|$.

Lemma 4.1. *If a function $f : \mathcal{G} \to \mathbb{R}_+$ defined on a finite connected graph $\mathcal{G}$ is (ℓ, q)-subharmonic in a ball $\mathrm{B}_L(x) \subsetneq \mathcal{G}$, with $L \geq \ell \geq 0$, then*

$$f(x) \leq q^{\left\lfloor \frac{L+1}{\ell+1} \right\rfloor} \mathcal{M}(f, \mathcal{G}) \leq q^{\frac{L-\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G}). \tag{4.2}$$

In fact, the factor $\mathcal{M}(f, \mathcal{G})$ in the RHS of (4.2) can be replaced by $\mathcal{M}(f, \mathrm{B}_{L+1}(x))$.

Proof. Since $L \geq \ell \geq 0$, we have $n + 1 := \left\lfloor \frac{L+1}{\ell+1} \right\rfloor \geq 1$. Set $\Lambda_j := \mathrm{B}_{j(\ell+1)}(x)$, $0 \leq j \leq n$, and note that $\Lambda_{n+1} \subset \mathrm{B}_{L+1}(x)$, since $(n+1)(\ell+1) \leq L+1$. Further, if $y \in \Lambda_j$ with $0 \leq j \leq n$ and $z \in \mathrm{B}_{\ell+1}(y)$, then, by triangle inequality, $z \in \Lambda_{j+1}$. Consider a monotone non-decreasing function $h : [0, L+1] \cap \mathbb{N} \mapsto \mathbb{R}_+$ defined by $h(r) = \mathcal{M}(f, \mathrm{B}_r(x))$. Using the (ℓ, q)-subharmonicity of f, we obtain

$$h(j(\ell+1)) \leq q \max_{y \in \Lambda_j} \max_{z \in \mathrm{B}_{\ell+1}(y)} f(z) \leq q h((j+1)(\ell+1)),$$

in particular,

$$h(n(\ell+1)) \leq q h(L+1) \leq q \mathcal{M}(f, \mathcal{G}).$$

Since $f(x) = h(0)$, the claim follows by the backward recursion in j from n to 0, in n steps. $\square$

Example. $\mathcal{G} = [0, R] \cap \mathbb{Z}$, $R = L+1$, $\ell = 0$, and $f : x \mapsto q^{R-x}$, $x \in \mathcal{G}$. For all $y \in \mathrm{B}_L(0) = [0, L]$, one has

$$f(x) = q^{R-x} = q \max_{|y-x| \leq 1} q^{R-y} = q \max_{|y-x| \leq 1} f(y),$$

which implies the $(0, q)$-subharmonicity of the function f in $\mathrm{B}_L(0)$. In fact, here the inequality of the form (4.1) turns out to be an equality, and one has

$$f(0) = q^{L+1} = q^{\frac{L+1}{0+1}} f(L+1),$$

which shows that the estimate from Lemma 4.1 is sharp. Note also that the inequality (4.1) cannot be extended to the exterior point $y = L+1$, since

$$f(L+1) = 1 > q = f(L).$$

Clearly, a function (ℓ, q)-subharmonic everywhere in $\mathcal{G}$, with $q < 1$, must be zero:

$$0 \leq \max_x f(x) \leq q \max_y f(y).$$

Naturally, the main *raison d'être* of the Definition 4.1 is the following fact.

Lemma 4.2. *Consider a finite connected graph $\mathcal{G}$ and a ball $\mathrm{B}_L(u) \subsetneq \mathcal{G}$, with $L \geq \ell \geq 0$. Fix numbers $E \in \mathbb{R}$, $m > 0$ and suppose that all balls $\mathrm{B}_\ell(x)$ inside $\mathrm{B}_L(u)$ are (E,m)-NS. Then $\forall\, y \in \mathcal{G} \setminus \mathrm{B}_L(u)$ the function*

$$f : x \mapsto |G_{\mathcal{G}}(x,y;E)|$$

is (ℓ,q)-subharmonic in $\mathrm{B}_L(u)$ with $q = \mathrm{e}^{-\gamma(m,\ell)\ell}$.

Proof. The claim follows directly from the Definition 4.1. □

Lemma 4.1 suffices to assess the Green functions in a ball $\mathrm{B}_L(u)$ which does not contain any singular ℓ-ball, but to analyze the situation where $\mathrm{B}_L(u)$ contains one singular ball $\mathrm{B}_\ell(w)$ (more precisely, it does not contain any pair of disjoint singular ℓ-balls), one needs the following extension of Lemma 4.1, which exploits the idea used of the proof of Theorem 1 in [26]: approaching a single "bad" ball separately from the points x and y.

Lemma 4.3. *Let $\mathcal{G}$ be a finite connected graph, and $f : \mathcal{G} \times \mathcal{G} \to \mathbb{R}_+$, $f : (x,y) \mapsto f(x,y)$, be a function which is separately (ℓ,q)-subharmonic in $x \in \mathrm{B}_{r'}(u') \subset \mathcal{G}$ and in $y \in \mathrm{B}_{r''}(u'') \subset \mathcal{G}$, with $r', r'' \geq \ell \geq 0$ and $\mathrm{d}(u',u'') \geq r' + r'' + 2$. Then*

$$f(u',u'') \leq q^{\left\lfloor \frac{r'+1}{\ell+1} \right\rfloor + \left\lfloor \frac{r''+1}{\ell+1} \right\rfloor} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}) \leq q^{\frac{r'+r''-2\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}). \tag{4.3}$$

Proof. For each $y'' \in \mathrm{B}_{r''+1}(u'')$ define the function $f_{y''} : x' \mapsto f(x',y'')$ in $\mathcal{G}$. By assumption, it is (ℓ,q)-subharmonic in $\mathrm{B}_{r'}(u')$, so Lemma 4.1 implies,

$$\forall\, y'' \in \mathrm{B}_{r''+1}(u'') \quad f(u',y'') = f_{y''}(u') \leq q^{\frac{r'-\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}).$$

Consider now another function, $\tilde{f}_{u'} : y'' \mapsto f(u',y'')$, $y'' \in \mathcal{G}$. It is (ℓ,q)-subharmonic in $\mathrm{B}_{r''}(u'')$, by hypothesis. The above inequality reads as

$$\mathcal{M}(\tilde{f}_{u'}, \mathrm{B}_{r''+1}(u'')) \leq q^{\frac{r'-\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}),$$

so another application of Lemma 4.1 proves the claim:

$$f(u',u'') = \tilde{f}_{u'}(u'') \leq q^{\frac{r'-\ell}{\ell+1}} \mathcal{M}(\tilde{f}_{u'}, \mathrm{B}_{r''+1}(u'')) \leq q^{\frac{r'+r''-2\ell}{\ell+1}} \mathcal{M}(f, \mathcal{G} \times \mathcal{G}). \quad \square$$

5. Fixed-energy scale induction

5.1. Scaling of Green functions in absence of tunneling.

Definition 5.1. A ball $\mathrm{B}_{L_{k+1}}(u)$ is called E-tunneling (E-T) if it contains two disjoint (E,m)-S balls of radius L_k, and E-non-tunneling (E-NT), otherwise.

Lemma 5.1. *If a ball $\mathrm{B}_{L_{k+1}}(u)$ is E-NR and E-NT, then it is (E,m)-NS.*

Proof. (See Fig. 1.) Fix two points $x, y \in \mathrm{B}_{L_{k+1}}(u)$ with $\mathrm{d}(x,y) \geq L_k^{1+\varrho} = L_k^{7/6}$ and let $R = \mathrm{d}(x,y)$, so $R - 2L_k \geq R(1 - 2L_k^{-\varrho})$. Since $\mathrm{B}_{L_{k+1}}(u)$ is E-NT, there is a ball $\mathrm{B}_{2L_k}(w)$ such that any L_k-ball disjoint with $\mathrm{B}_{2L_k}(w)$ is (E,m)-NS. By triangle inequality, there are integers $r', r'' \geq 0$ such that $r' + r'' \geq R - 2L_k$, the balls $\mathrm{B}_{r'}(x)$ and $\mathrm{B}_{r''}(y)$ are disjoint from each other and from $\mathrm{B}_{2L_k}(w)$, so any ball $\mathrm{B}_{L_k}(v)$ inside $\mathrm{B}_{r'}(x)$ and inside $\mathrm{B}_{r''}(y)$ is (E,m)-NS.

Assume first that $r' \geq L_k$ and $r'' \geq L_k$ (otherwise, one of the points x, y is covered by $\mathrm{B}_{2L_k}(w)$, so one of the radii $r', r'' \geq R - 3L_k$ and the same argument

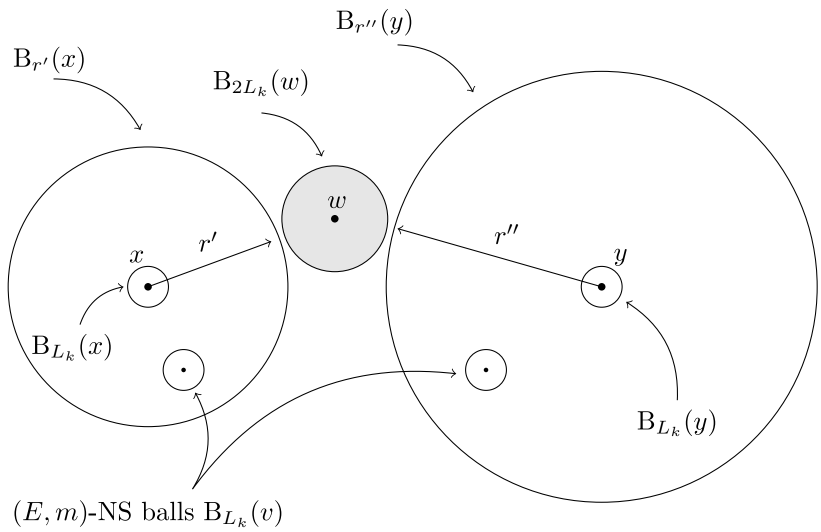

FIGURE 1. Example for the proof of Lemma 5.1.

as below applies). By Lemma 4.2, the function $f : (v,z) \mapsto |G_{\mathrm{B}_{L_{k+1}}}(v,z;E)|$ is (L_k,q)-subharmonic in $v \in \mathrm{B}_{r'}(x)$ and in $z \in \mathrm{B}_{r''}(y)$, with $q \le \mathrm{e}^{-mR(1+L_k^{-\tau})}$. By Lemma 4.3, one can write, with the convention $-\ln 0 = +\infty$:

$$
\begin{aligned}
-\ln f(x,y) &\ge -\ln\left\{ \left(e^{-m(1+L_k^{-\tau})L_k}\right)^{\frac{R\left(1-2L_k^{-\varrho}\right)}{L_k+1}} e^{L_k^{\beta}} \right\} \\
&\ge mR\left(\left(1+\tfrac{1}{2}L_k^{-\tau}\right) \tfrac{L_k}{L_k+1} \left(1-2L_k^{-\varrho}\right) - L_k^{1-\beta} m^{-1} R^{-1} \right)
\end{aligned}
$$

(with $m \ge 1$, $R \ge L_k^{1+\varrho} = \mathrm{d}(x,y) \ge L_k^{7/6}$ $\beta = 1/2$ and L_0 large enough)

$$
\begin{aligned}
&\ge mR\left((1+L_k^{-1/8})\big(1-3L_k^{-1/6}\big) - L_k^{-2/3} \right) \\
&\ge m\mathrm{d}(x,y)\left(1+\tfrac{1}{2}L_k^{-1/8}\right) \\
&\ge \gamma(m,L_{k+1})\,\mathrm{d}(x,y) + \ln|\partial \mathrm{B}_{L_{k+1}}(u)|,
\end{aligned}
$$

as required for the (E,m)-NS property of the ball $\mathrm{B}_{L_{k+1}}(u)$. □

5.2. **Scale induction.** Introduce the following notations:

$$
\begin{aligned}
P_k &= \sup_{x\in\mathcal{Z}} \mathbb{P}\left\{ \mathrm{B}_{L_k}(x) \text{ is } (E,m)\text{-S} \right\}, \\
Q_k &= 2\sup_{x\in\mathcal{Z}} \mathbb{P}\left\{ \mathrm{B}_{L_k}(x) \text{ is } E\text{-R} \right\} \le 2C_W\, C_d L_k^d\, \mathrm{e}^{-L_k^{\beta}}
\end{aligned}
$$

(the latter inequality uses the Wegner estimate (2.8)).

Theorem 5.2. *If there is an integer $L_0 \geq 1$ such that* (2.7) *is fulfilled and*

$$\min\{P_0, Q_0\} \leq C_d^{-2} L_0^{-\kappa},\ \kappa \geq \tfrac{2\alpha d}{2-\alpha},$$

then for all $k \geq 0$, $P_k \leq C_d^{-2}\, L_k^{-\kappa}$.

Proof. (The main argument combines the ideas from [26] and [20].) By virtue of Lemma 5.1, if a ball $\mathrm{B}_{L_{k+1}}(u)$ is (E,m)-S, then it is either E-R or E-T. There are $< \frac{1}{2} C_d^2 L_{k+1}^{2d}$ pairs of disjoint L_k-balls in $\mathrm{B}_{L_{k+1}}(u)$, thus

$$P_{k+1} \leq \frac{1}{2} C_d^2 L_{k+1}^{2d} \mathcal{P}_k^2 + \frac{1}{2} Q_{k+1}.$$

By Wegner estimate (2.8), $Q_{k+1} \leq C_W C_d L_{k+1}^d \mathrm{e}^{-L_{k+1}^{1/2}}$. An elementary calculation shows that the function

$$f : L \mapsto \ln\left(\mathrm{const}\, L^{-\kappa}\right) - \ln\left(\mathrm{const}\ L^d \mathrm{e}^{-L^{1/2}}\right) = L^{1/2} - \mathrm{const} \ln L$$

on $[1, +\infty)$ is either non-negative or admits a unique zero. In either case, the assumption $Q_0 \leq C_d^{-2} L_0^{-\kappa}$ implies $Q_{k+1} \leq C_d^{-2} L_{k+1}^{-\kappa}$ for all $k \geq 0$. Thus by induction on $k = 0, 1, \ldots,$

$$P_{k+1} \leq 2 \cdot \frac{1}{2} C_d^2 L_{k+1}^{2d} P_k^2 \leq \frac{C_d^2}{\left(C_d^2\right)^2} L_{k+1}^{-\frac{2\kappa}{\alpha} + 2d} \leq C_d^{-2} L_{k+1}^{-\kappa}.$$

provided that $\frac{2\kappa}{\alpha} + 2d \geq \kappa$, i.e., for all $\kappa \geq \frac{2\alpha d}{2-\alpha}$. □

This calculation shows that the power-law bound on the probabilities $\mathcal{P}_k$ is reproduced at each scale L_k with the same exponent $\kappa > 0$. Actually, it comes with a small bonus which seems to have never been used, until recently. Pick a value $\kappa > \frac{2\alpha d}{2-\alpha}$, so that $\theta := \left(\frac{2}{\alpha} - \frac{2d}{\kappa}\right) - 1 > 0$, and observe that

$$\tfrac{2\kappa(1+\theta)^k}{\alpha} - 2d \geq \tfrac{2\kappa(1+\theta)^k}{\alpha} - 2d = \kappa(1+\theta)^k \left\{\tfrac{2}{\alpha} - \tfrac{2d}{\kappa}\right\} = \kappa(1+\theta)^{k+1}.$$

If for some $k_0 \geq 0$ one has $P_{k_0} \leq C_d^{-2} L_{k_0}^{-\kappa(1+\theta)^{k_0}}$, then by induction, for all $k \geq k_0$,

$$\begin{aligned} P_{k+1} &\leq 2 \cdot \frac{1}{2} C_d L_{k+1}^{2d} P_k^2 \leq \frac{C_d^2}{C_d^4} L_{k+1}^{-\frac{2\kappa(1+\theta)^k}{\alpha} + 2d} \\ &\leq C_d^{-2} L_{k+1}^{-\kappa(1+\theta)^{k+1}}. \end{aligned}$$

This gives rise to the probabilities P_k decaying *faster that any power law.*

Observe also that taking $\alpha \downarrow 1$ results in $2\alpha/(2-\alpha) \downarrow 2$, so that the exponent $\kappa > 0$ in the *hypothesis* of Lemma 5.2 can be chosen arbitrarily close to $2d$.

We see that the complete, fixed-energy MSA procedure can be effectively reduced to Lemma 5.1 and Theorem 5.2 and results in upper bounds on the probability of singular balls decaying faster than any power-law.

6. From fixed to variable energy: First approach

Now we establish a fairly general relation between fixed-energy probabilistic estimates on the Green functions and variable-energy bounds for two disjoint finite volumes. It does not matter how the probabilistic input is obtained; in particular, the results of this section can be combined both with the MSA, performed for each

fixed energy E in a given interval $I \subset \mathbb{R}$, and with the FMM (which always starts as a fixed-energy analysis).

It is convenient to assume that $|I| = 1$, so the interval I with the Lebesgue measure $\mathrm{mes}(\cdot)$ is a probability space, and so is the product space $(\Omega \times I, \mathbb{P} \times \mathrm{mes})$. (The idea of using the "disorder-energy" space with product measure has been used earlier, e.g., in [23] and [28].) Given $L \in \mathbb{N}$ and points $x, y \in \mathcal{Z}$, set for brevity

$$\mathcal{M}_{x,y}(E) = |G_{\mathrm{B}_L}(x, y; E)|, \ \mathcal{M}_x(E) = \max_{y \in \partial^- \mathrm{B}_L(x)} \mathcal{M}_{x,y}(E), \tag{6.1}$$

and introduce the subsets of I parameterized by $a > 0$:

$$\mathcal{E}_{x,y}(a) = \{E \in I : \mathcal{M}_{x,y}(E) \geq a\}, \ \ \mathcal{E}_x(a) = \{E \in I : \mathcal{M}_x(E) \geq a\}. \tag{6.2}$$

(The L-dependence will be often omitted for brevity.)

Theorem 6.1. *Let $L \geq 0$, $x \in \mathcal{Z}$, $y \in \partial^- \mathrm{B}_L(x)$. Let $\{\lambda_j\}_{j=1}^N$ be the eigenvalues of the operator $H_{\mathrm{B}_L(u)}(\omega)$ and $I \subset \mathbb{R}$ an interval of unit length. Let be given numbers $a, b, c, \mathcal{P}_L > 0$ such that*

$$b \leq \min \left\{ |\mathrm{B}_L(u)|^{-1} a c^2, \, c \right\}, \tag{6.3}$$

and for all $E \in I$

$$\mathbb{P} \left\{ \mathcal{E}_x(a) \right\} \equiv \mathbb{P} \left\{ \mathcal{M}_x(E) \geq a \right\} \leq \mathcal{P}_L. \tag{6.4}$$

There is an event $\mathcal{B}_x(b)$ with $\mathbb{P} \{ \mathcal{B}_x(b) \} \leq b^{-1} \mathcal{P}_L$ such that $\forall \ \omega \notin \mathcal{B}_x(b)$, the set

$$\mathcal{E}_x(2a) = \mathcal{E}_x(2a; \omega) = \Big\{ E : \mathcal{M}_x(E) \geq 2a \Big\}$$

is contained in a union of intervals $\cup_{j=1}^N I_j$, $I_j := \{E : |E - \lambda_j| \leq c\}$, $\lambda_j \in I$.

Proof. Consider the following events parameterized by $b > 0$:

$$\mathcal{B}_x(b) = \{ \, \omega \in \Omega : \, \mathrm{mes}(\mathcal{E}_x(a)) > b \, \}. \tag{6.5}$$

Apply Chebyshev's inequality and the Fubini theorem combined with (6.4):

$$\begin{aligned} \mathbb{P} \left\{ \mathcal{B}_x(b) \right\} &\leq b^{-1} \mathbb{E} \left[\, \mathrm{mes}(\mathcal{E}_x(a)) \, \right] \\ &= b^{-1} \int_I dE \; \mathbb{E} \left[\, \mathbf{1}_{\{\mathcal{M}_x(E) \geq a\}} \, \right] \leq b^{-1} \mathcal{P}(L). \end{aligned} \tag{6.6}$$

Now fix any $\omega \notin \mathcal{B}_x(b)$, so that $\mathrm{mes}(\mathcal{E}_x(a); \omega) \leq b$. There is a subset $\{\lambda_j\}_{j=1}^{N'}$ of the EVs of the operator $H_{\mathrm{B}_L(x)}$ such that the Green function $E \mapsto G_{\mathrm{B}_L(x)}(x, y; E)$ reads as a rational function (below we remove the vanishing terms, if any)

$$f : E \mapsto G_{\mathrm{B}_L(x)}(x, y; E) =: \sum_{j=1}^{N'} \frac{\kappa_j}{\lambda_j - E}, \ \ N' \leq N := |\mathrm{B}_L(x)|; \tag{6.7}$$

here $\kappa_j = \kappa_j(x, y) \neq 0$ and $\sum_j |\kappa_j| \leq \sum_i |\psi_i(x) \psi_i(y)| \leq N$. Let

$$\begin{aligned} \mathcal{R}(2c) &= \big\{ \lambda \in \mathbb{R} : \, \min_j |\lambda_j - \lambda| \geq 2c \big\}, \\ \mathcal{R}(c) &= \big\{ \lambda \in \mathbb{R} : \, \min_j |\lambda_j - \lambda| \geq c \big\}, \quad c > 0. \end{aligned}$$

Observe that, with $0 < b \leq c$, $\mathcal{A}_b := \{E : \, \mathrm{dist}(E, \mathcal{R}(2c)) < b\} \subset \mathcal{R}(c)$, hence, the set $\mathcal{A}_b$ is a union of open sub-intervals at distance $\geq c$ from the spectrum, and on each sub-interval one has $|f'(E)| \leq N c^{-2}$. Let us show by contraposition that, with $\omega \notin \mathcal{B}_{x,y}(b)$,

$$\{ \, E : \, |G_{\mathrm{B}_L}(x, y; E)| \geq 2a \, \} \cap \mathcal{R}(c) = \varnothing.$$

Assume otherwise, pick any point λ^* in the non-empty set in the LHS, and let $J := \{E' : |E' - \lambda^*| \le b\} \subset \mathcal{A}_b \subset \mathcal{R}(c)$. Then for any $E \in J$ one has, by (6.3),

$$|f(E)| \ge |f(\lambda^*)| - |J| \sup_{E' \in J} |f'(E')| > 2a - Nc^{-2} \cdot b \ge a,$$

so $J \subset \mathcal{E}_{x,y}(a)$ and $\mathrm{mes}(\mathcal{E}_{x,y}(a)) \ge \mathrm{mes}(J) = 2b > b$, contrary to the choice of ω. Since the set $\mathcal{R}(c)$ is independent of y, the assertion follows from (6.6). $\Box$

Below we provide some possible choices of the parameters a, b, c (depending, of course, upon the scale L) in three most frequently used frameworks.

(1) Weaker MSA-type bounds: $\mathcal{P}_{L_k} = P_k \le L_k^{-\kappa(1+\theta)^k}$, $\kappa > \frac{\alpha d}{2-\alpha}$, $\alpha = 3/2$. One can set, for $L \in \{L_k, k \ge 0\}$,

$$a(L_k) = L_k^{-\frac{3\kappa}{5}(1+\theta)^k}, \;\; b(L_k) = L_k^{-\frac{\kappa}{5}(1+\theta)^k}, \;\; c(L_k) = L_k^{-\left(\frac{\kappa}{5} - \frac{d}{2}\right)(1+\theta)^k}$$

(2) Sub-exponential MSA-type bounds: $\mathcal{P}_{L_k} \le \mathrm{e}^{-L_k^\delta}$, $\delta > 0$. Then one can set

$$a(L_k) = e^{-\frac{1}{3}L_k^\delta}, \;\; b(L_k) = e^{-\frac{2}{3}L_k^\delta}, \;\; c(L_k) = \mathrm{e}^{-\frac{1}{8}L_k^\delta}.$$

(3) FMM-type bounds: $\mathcal{P}_L \le \mathrm{e}^{-mL}$, $m > 0$. Then one can set, for all $L \in \mathbb{N}$ large enough,

$$a(L) = e^{-\frac{1}{3}mL}, \;\; b(L) = e^{-\frac{2}{3}mL}, \;\; c(L) = \mathrm{e}^{-\frac{m}{8}L}.$$

Theorem 6.2. *Assume the condition* (2.7). *Suppose that for some $L \in \mathbb{N}$, numbers $a = a(L)$, $b = b(L)$, $c = c(L)$ and $\mathcal{P}_L > 0$ obey* (6.3), *and for some interval $I \subset \mathbb{R}$ and all $E \in I$, for any ball $\mathrm{B}_L(x) \subset \mathcal{Z}$*

$$\mathbb{P}\{\, \mathcal{M}_x(E) \ge a \,\} \le \mathcal{P}_L. \tag{6.8}$$

Then for any two disjoint balls $\mathrm{B}_L(x)$, $\mathrm{B}_L(y) \subset \mathcal{Z}$ the following bound holds true:

$$\mathbb{P}\{\, \exists\, E \in I : \min(\mathcal{M}_x(E), \mathcal{M}_y(E)) > a(L) \,\} \le 4C_W C_d^2 L^{2d}\, c(L) + \frac{2\mathcal{P}_L}{b(L)}. \tag{6.9}$$

Proof. Let the events $\mathcal{B}_x(b)$, $\mathcal{B}_y(b)$ be defined as in (6.5) and $\mathcal{B} = \mathcal{B}_x \cup \mathcal{B}_y$, then

$$\begin{aligned} \mathbb{P}\{\, \mathcal{E}_x(a) \cap \mathcal{E}_y(a) \ne \varnothing \,\} &\le \mathbb{P}\{\, \mathcal{B} \,\} + \mathbb{E}\left[\mathbb{P}\{\, \{\mathcal{E}_x(a) \cap \mathcal{E}_y(a) \ne \varnothing\} \cap \mathcal{B}^{\mathrm{c}} \,\} \right] \\ &\le 2b^{-1}\mathcal{P}_L + \mathbb{E}\left[\mathbb{P}\left\{\, \{\mathcal{E}_x(a) \cap \mathcal{E}_y(a) \ne \varnothing\} \cap \mathcal{B}^{\mathrm{c}} \,|\, \mathcal{F}_{\mathrm{B}_L(y)} \,\right\} \right]. \end{aligned} \tag{6.10}$$

It remains to assess the conditional probability in the RHS. For $\omega \notin \mathcal{B}^{\mathrm{c}}$, each of the sets $\mathcal{E}_x(a)$, $\mathcal{E}_y(a)$ is covered by intervals of width $2c(L)$ around the respective EVs $\lambda_i(x) \in \Sigma(H_{\mathrm{B}_L(x)}$, $\lambda_j(y) \in \Sigma(H_{\mathrm{B}_L(y)}$, and for disjoint balls $\mathrm{B}_L(x), \mathrm{B}_L(y)$ these spectra are independent. Now apply Theorem 6.1 and the Wegner estimate:

$$\begin{aligned} \mathbb{P}\left\{\, \{\mathcal{E}_x(a) \cap \mathcal{E}_y(a) \ne \varnothing\} \cap \mathcal{B}^{\mathrm{c}} \,|\, \mathcal{F}_{\mathrm{B}_L(y)} \,\right\} &\le |\mathrm{B}_L(y)| \sup_{\lambda \in I} \mathbb{P}\{\, \mathrm{dist}(\mathcal{E}_x(a), \lambda) \le c \,\} \\ &\le |\mathrm{B}_L(y)| \sup_{\lambda \in I} \mathbb{P}\left\{\, \mathrm{dist}(\Sigma(H_{\mathrm{B}_L(x)}, \lambda) \le 2c \,\right\} \\ &\le 4C_W |\mathrm{B}_L(y)| \cdot |\mathrm{B}_L(x)| \cdot c(L). \end{aligned} \tag{6.11}$$

Collecting (6.10) and (6.11), the assertion follows. $\Box$

It is clear that the above approach, albeit very general and based on a Wegner-type bound, gives rise to exponential decay bounds on the Green functions only if the fixed-energy analysis provides exponential probabilistic bounds; as it is well-known, this has been achieved so far only in the framework of the FMM.

6.1. **Spectral localization.** The assertion of Theorem 7.3 has a structure similar to that of the MSA bound from the work by von Dreifus and Klein [12]. More precisely, it guarantees a decay rate of Green functions slower than exponential, but faster than any power-law. It is not difficult to adapt the well-known argument from [12] and prove that with probability one, all polynomially bounded solutions to the equation $H(\omega)\psi = E\psi$ are in fact square-summable. The latter property requires a Shnol–Simon type result on spectrally a.e. polynomial boundedness of generalized eigenfunctions. It will follow independently by RAGE (Ruelle–Amrein–Georgescu–Enss) theorems (see a detailed discussion along with a bibliography, e.g., in [9]) from the dynamical localization proven in Section 8.

7. From fixed to variable energy: Second approach

7.1. The spectral reduction.

Theorem 7.1. *Let be given a ball* $\mathrm{B}_L(x)$*,* $L \geq 1$*, and numbers* $a(L)$*,* $b(L)$*,* $c(L)$*,* $\mathcal{P}_L > 0$ *obeying* (6.3) *and such that, for some interval* I*, all* $E \in I$*,*

$$\mathbb{P}\left\{ \mathcal{M}_x(E) \geq a \right\} \leq \mathcal{P}_L. \tag{7.1}$$

Then the following properties hold true:

(A) *For any* $b \geq \mathcal{P}_L$ *there exists an event* $\mathcal{B}_b$ *such that* $\mathbb{P}\left\{ \mathcal{B}_b \right\} \leq b^{-1}\mathcal{P}_L$ *and for any* $\omega \notin \mathcal{B}_p$ *the set of energies*

$$\mathcal{E}_x(a) = \mathcal{E}_x(a;\omega) = \{\mathcal{M}_x(E) \geq a\} \cap I$$

is covered by $K < 3N^2$ *intervals* $J_i = [E_i^-, E_i^+]$ *of total length* $\sum_i |J_i| \leq b$*.*

(B) *The endpoints* $E_i^{\pm}$ *are determined by the function* $E \mapsto \langle \mathbf{1}_x \mid (H_{\mathbf{B}_L(u)} - E)^{-1} \mid \mathbf{1}_y\rangle$ *in such a way that, for the one-parameter family* $A(t) := H_{\mathrm{B}_L(u)} + t\,\mathbf{1}$*, the endpoints* $E_i^{\pm}(t)$ *for the operators* $A(t)$ *(replacing operators* $H_{\mathrm{B}_L(u)}$*) have the form*

$$E_i^{\pm}(t) = E_i^{\pm}(0) + t, \quad t \in \mathbb{R}.$$

Proof. (A) Fix a point $y \in \partial^-\mathrm{B}_L(x)$ and consider the rational function

$$f_y : E \mapsto \sum_{i=1}^N \frac{\kappa_i}{\lambda_i - E} := \sum_{i=1}^N \frac{\psi_i(x)\,\psi_i(y)}{\lambda_i - E}.$$

Its derivative has the form

$$f_y'(E) = \sum_{i=1}^N \frac{-\kappa_i}{(\lambda_i - E)^2} =: \frac{\mathscr{P}(E)}{\mathscr{Q}(E)}, \;\; \deg \mathscr{P} \leq 2N-2,$$

and has $\leq 2N-2$ zeros and $\leq N$ poles, so f_y has $\leq 3N-1$ intervals of monotonicity $I_{i,y}$, and the total number of monotonicity intervals of all functions $\{f_y, y \in \partial^-\mathrm{B}_L(x)\}$ is bounded by $K \leq |\partial^-\mathrm{B}_L(x)|(3N-1) \leq |\mathrm{B}_L(x)|(3N-1) < 3N^2$, so

$$\cup_{y\in\partial^-\mathrm{B}_L(x)}\{E :\; f_y(E) \geq a\} = \cup_{i=1}^K J_i, \quad J_i = [E_i^-, E_i^+] \subset I,$$

where, obviously, $\sum_i |J_i| \leq \mathrm{mes}\,\{E :\; \mathcal{M}_x(E) \geq a\}$.

(B) Consider a one-parameter operator family $A(t) = H_{\mathrm{B}_L(u)}(\omega) + t\,\mathbf{1}$. All these operators share common eigenvectors; the latter determine the coefficients κ_i, so one can choose eigenfunctions $\psi_i(t)$ constant in t and obtain $\kappa_i(t) = \kappa_i(0)$. The eigenvalues of operators $A(t)$ have the form $\lambda_i(t) = \lambda_i(0) + t$. We conclude that the

Green functions, with fixed x and y, have the form $f_{x,y}(E;t) = f_{x,y}(E-t;0)$, so that the intervals $J_i(t)$ have indeed the form $J_i(t) = [E_i^- + t, E_i^+ + t]$. $\square$

Theorem 7.2. *Consider two disjoint balls* $\mathrm{B}_L(x)$, $\mathrm{B}_L(y)$ *and the random variables*

$$\xi_x(\omega) := |\mathrm{B}_L(x)|^{-1} \sum_{z\in \mathrm{B}_L(x)} V(z;\omega), \quad \eta_z(\omega) := V(z;\omega) - \xi_x(\omega),\ z \in \mathrm{B}_L(x)$$

(the sample average and fluctuations of the potential in $\mathrm{B}_L(x)$*). Let* $\mathfrak{F}_x$ *be the sigma-algebra generated by the random variables* $\{\eta_y, y \in \mathrm{B}_L(y); V(z;\cdot), z \notin \mathrm{B}_L(x)\}$. *Consider the conditional probability distribution function*

$$F_{\xi_x}(t \,|\, \mathfrak{F}_x) = \mathbb{P}\left\{ \xi_x \le t \,|\, \mathfrak{F}_x \right\}$$

and its continuity modulus

$$\nu_{\xi_x}(s \,|\, \mathfrak{F}_x) = \sup_{t\in\mathbb{R}} \operatorname{ess\,sup} \left(F_{\xi_x}(t+s \,|\, \mathfrak{F}_x) - F_{\xi_x}(t \,|\, \mathfrak{F}_x) \right).$$

Suppose that, for some $C, C', A, A', B, B' \in (0,+\infty)$

$$\forall\, s \in [0,1] \qquad \mathbb{P}\left\{ \nu_{\xi_x}(s \,|\, \mathfrak{F}_x) > CL^A s^B \right\} \le C' L^{A'} s^{B'}. \tag{7.2}$$

Then

$$\mathbb{P}\left\{ \exists\, E \in I:\ \min\{\mathcal{M}_x(E), \mathcal{M}_y(E)\} \ge a \right\} \le N^2 \widetilde{h}(4b) \tag{7.3}$$

where

$$\widetilde{h}(s) := CL^A s^B + C' L^{A'} s^{B'}.$$

Proof. Using the decomposition $V(z;\omega) = \xi_x(\omega)\,\mathbf{1} + \eta_z(\omega)$ in the ball $\mathrm{B}_L(x)$, , consider the respective operator decomposition

$$H_{\mathrm{B}_L(x)}(\omega) = A_x(\omega) + \xi_x(\omega)\,\mathbf{1}, \quad A(\omega) = H_0 + \eta_\bullet(\omega),$$

where, conditional on $\mathfrak{F}_x$, the operator $A(\omega)$ is non-random. Fix a number $b > 0$ and consider the events $\mathcal{B}_b(x)$ (relative to the operator $H_{\mathrm{B}_L(x)}$) and, respectively, $\mathcal{B}_b(y)$; let $\mathcal{B} = \mathcal{B}_b(x) \cup \mathcal{B}_b(y)$. For any $\omega \notin \mathcal{B}$, the energies E where $\mathcal{M}_x(E) \ge a$ are covered by intervals $J_{i,x} = [E_{i,x}^-, E_{i,x}^+]$, with $\sum_i |J_{i,x}| \le b$, and, respectively, the energies E where $\mathcal{M}_y(E) \ge a$ are covered by intervals $J_{i,y} = [E_{i,y}^-, E_{i,y}^+]$, also obeying $\sum_i |J_{i,y}| \le b$. Conditional on $\mathfrak{F}_x$, intervals $J_{i,y}$ become non-random, while for the intervals $J_{i,x}$ we can write, by virtue of assertion (B) of Theorem 7.1,

$$J_{i,y}(\omega) = [E_{i,x}^{(-,0)}(\omega) + \xi_x(\omega),\ E_{i,x}^{(+,0)}(\omega) + \xi_x(\omega)]$$

where $E_{i,x}^{(\pm,0)}(\omega)$ are $\mathfrak{F}_x$-measurable, i.e., non-random under the conditioning by $\mathfrak{F}_x$.

Further, set $\epsilon_{i,x} = |J_{i,x}|$ and $\epsilon_{j,y} = |J_{j,y}|$, then

$$\begin{aligned} \{\omega:\ J_{i,x} \cap J_{j,y} \ne \varnothing\} &\subset \left\{ \left| E_{i,x}^{(-,0)}(\omega) - E_{j,y}^{(-,0)}(\omega) \right| \le \epsilon_{i,x} + \epsilon_{j,y} \right\} \\ &= \left\{ \left| \xi_x(\omega) - \widetilde{E}(\omega) \right| \le \epsilon_{i,x} + \epsilon_{j,y} \right\} \end{aligned}$$

with $\widetilde{E}(\omega)$ non-random under the conditioning. Apply the assumption (7.2):

$$\begin{aligned} \mathbb{P}\left\{ \left| \xi_x(\omega) - \widetilde{E}(\omega) \right| \le \epsilon_{i,x} + \epsilon_{j,y} \right\} &\le \mathbb{P}\left\{ \left| \xi_x(\omega) - \widetilde{E}(\omega) \right| \le 4b \right\} \\ &\le \mathbb{P}\left\{ \nu_{\xi_x}(4b \,|\, \mathfrak{F}_x) > CL^A (4b)^B \right\} + CL^A (4b)^B \\ &= \widetilde{h}(4b). \end{aligned}$$

Taking the sum over i, j, we obtain the erquired bound:

$$\begin{aligned}\mathbb{P}\{\, \exists E \in I:\ \min\{\mathcal{M}_x(E), \mathcal{M}_y(E)\} \geq a \,\} &\leq \sum_{i,j} \mathbb{P}\{\, \omega:\ J_{i,x} \cap J_{j,y} \neq \varnothing \,\} \\ &\leq N^2 \widetilde{h}(4b). \quad \square\end{aligned}$$

In particular, taking into account Theorem 5.2, we can set, for $L = L_k$,

$$a = a(L_k) = \mathrm{e}^{-\gamma(m,L_k)L_k},\ \mathcal{P}_{L_k} = L_k^{-\kappa(1+\theta)^k},\ b = b(L_k) = L_k^{-\frac{\kappa}{2}(1+\theta)^k}.$$

These settings give rise to the following corollary of Theorem 7.2:

Theorem 7.3. *If there is an integer $L_0 \geq 1$ and numbers $m \geq 1$, $\alpha \in (1, 2)$ such that (2.7) is fulfilled and*

$$\min\{P_0, Q_0\} \leq C_d^{-2} L_0^{-\kappa},\ \kappa > \tfrac{2\alpha d}{2-\alpha},$$

then for some $\theta > 0$ and all $k \geq 0$, for any interval $I \subset \mathbb{R}$ with $|I| \leq 1$,

$$\mathbb{P}\{\, E \in I:\ \mathrm{B}_L(x) \text{ and } \mathrm{B}_L(y) \text{ are } (E, m)\text{-}S \,\} \leq L_k^{-\frac{\kappa}{2}(1+\theta)^k}.$$

7.2. **On the validity of the assumption** (7.2)**.** First of all, recall that, by an elementary result on Gaussian distributions, if $V : \mathcal{Z} \times \Omega \to \mathbb{R}$ is an IID Gaussian field, say, with zero mean and unit variance, the sample average ξ_x of the sample $\{V(z;\omega), z \in \mathrm{B}_L(x)\}$ is independent of the sigma-algebra generated by the "fluctuations" $\eta_z(\omega)$; moreover, it has Gaussian distribution $\mathcal{N}(0, |\mathrm{B}_L(x)|)$ and admits a probability density with $\|p_{\xi_x}\|_\infty \leq \frac{1}{\sqrt{2\pi}} |\mathrm{B}_L(x)|^{1/2}$. In this particular case, Eqn (7.2) can be replaced by a stronger, deterministic bound: the conditional continuity modulus $\nu_{\xi_x}(s \,|\, \mathfrak{F}_x)$ is actually independent of the condition and is bounded by $\|p_{\xi_x}\|_\infty \cdot s$.

Such a situation is rather exceptional, as shows the example of two IID random variables $V_1(\omega), V_2(\omega)$ with uniform distribution $\mathrm{Unif}([0, 1])$. Indeed, in this case $\xi := (V_1{+}V_2)/2$, $\eta = (V_1{-}V_2)/2$ and the distribution of ξ conditional on η is uniform on the interval I_η of length $O(1{-}|\eta|)$, hence, with constant density $O(\big|1{-}|\eta|\big|^{-1})$, for $|\eta| < 1$; for $\eta = \pm 1$, this distribution is concentrated on a single point. However, this example shows also how such a difficulty can be bypassed: excessively "singular" conditional distrubutions of the sample mean ξ occur only for a set of conditions having a small probability. Using this simple idea, Gaume [18], in the framework of his PhD project, established the property (7.2) for IID random fields with piecewise constant marginal *probability density*. By standard approximation arguments, it can be easily extended to piecewise Lipshitz (or Hölder) continuous *densities*, which is sufficient for most physically relevant applications. We believe that some variant of the property (7.2), perhaps weaker but still sufficient for the purposes of the MSA, holds true in a larger class of IID random fields.

7.3. **Exponential spectral localization.** The assertion of Theorem 7.3 has the same form as in the conventional MSA bound going back to the work by von Dreifus and Klein [12] (actually, even slightly stronger); therefore, the same argument as in [12] (having its roots in [17]) applies and proves that with probability one, all polynomially bounded solutions to the equation $H(\omega)\psi = E\psi$ are in fact decaying exponentially fast at infinity, thus the operator $H(\omega)$ has a.s. pure point spectrum. The latter property requires a Shnol–Simon type result on spectrally a.e. polynomial

boundedness of generalized eigenfunctions; it will follow by RAGE theorems from the dynamical localization proven in Section 8.

8. From MSA to strong dynamical localization

The first rigorous derivations of the dynamical localization from MSA-type probabilistic bounds on the Green functions have been obtained by Germinet–De Bièvre [19] and Damanik–Stollmann [13]. The latter paper had a very eloquent title: "*Multiscale analysis implies strong dynamical localization*". One of the main ingredient of these two works is the analysis of the so-called centers of localization of square-summable eigenfunctions; this notion appeared earlier in the work [11] which proved instrumental for a number of subsequent researches. Later, Germinet and Klein [20] discovered a substantially shorter argument, using more efficiently Hilbert–Schmidt norm estimates for spectral projections in a infinitely extended configuration space. Formally, [20] considers operators in a Euclidean space $\mathbb{R}^d$, but an adaptation to a finite-dimensional lattice or, more generally, to a countable graph with polynomially growing balls, is quite straightforward.

In the present paper, working with finite graphs, we bypass the 'hard' analysis of spectral projections and replace it by Bessel's inequality.

The main result of this section can be summarized in the following meta-theorem, expressing the 'soft' argument by Germinet–Klein (viz., the finite-volume version thereof): "*The MSA bounds are essentially equivalent to the strong dynamical localization*", with a meta-proof: "*Apply Bessel's inequality*".

Owing to the results of Sections 6–7, it actually suffices to perform only the fixed-energy MSA, even in its simplest form proposed in [26].

The extension to an infinite configuration space also admits a short meta-proof, going back to the works by Aizenman *et al.*: "*Apply Fatou lemma*".

8.1. **EF correlators in finite balls.**

Given an interval $I \subset \mathbb{R}$, denote by $\mathscr{B}_1(I)$ the set of all Borel functions $\phi : \mathbb{R} \to \mathbb{C}$ with $\operatorname{supp} \phi \subset I$ and $\|\phi\|_\infty \le 1$.

Theorem 8.1. *Fix an integer $L \in \mathbb{N}^*$ and assume that the following bound holds for any pair of disjoint balls $\mathrm{B}_L(x), \mathrm{B}_L(y)$ and some quantity $\zeta(L) > 0$:*

$$\mathbb{P}\{\, \exists\, E \in I : \;\; \mathrm{B}_L(x) \text{ and } \mathrm{B}_L(y) \text{ are } (E,m)\text{-S}\,\} \le \zeta(L).$$

Then for any $x, y \in \mathcal{Z}$ with $\mathrm{d}(x,y) > 2L+1$, any finite connected subgraph (of $\mathcal{Z}$) $\mathcal{G} \supset \mathrm{B}_L(x) \cup \mathrm{B}_L(y)$ and any Borel function $\phi \in \mathscr{B}_1(I)$

$$\mathbb{E}\left[\, \left| \langle \mathbf{1}_x \,|\phi(H_{\mathcal{G}}(\omega))|\, \mathbf{1}_y \rangle \right| \,\right] \le 4\mathrm{e}^{-mL} + \zeta(L). \tag{8.1}$$

Proof. Fix points $x, y \in \mathcal{Z}$ with $\mathrm{d}(x,y) > 2L+1$ and a graph $\mathcal{G} \supset \mathrm{B}_L(x) \cup \mathrm{B}_L(y)$. The operator $H_{\mathcal{G}}(\omega)$ has a finite orthonormal eigenbasis $\{\psi_i\}$ with respective eigenvalues $\{\lambda_i\}$. Let $\mathbf{S} = \partial \mathrm{B}_L(x) \cup \partial \mathrm{B}_L(y)$ (recall: this is a set of *pairs* (u, u')); note that $|\mathbf{S}| \le 2C_d^2 L^d$, by (2.1). Suppose that for some ω, for each i there is $z_i \in \{x, y\}$ such that $\mathrm{B}_L(z_i)$ is (λ_i, m)-NS; let $\{v_i\} = \{x, y\} \setminus \{z_i\}$. Denote

$\mu_{x,y}(\phi) = \left| \langle \mathbf{1}_x \, | \phi(H_{\mathcal{G}}(\omega)) | \, \mathbf{1}_y \rangle \right|$, with $\mu_{x,y}(\phi) \le 1$. Then by the GRI for the eigenfunctions,

$$\begin{aligned}
\mu_{x,y}(\phi) &\le \|\phi\|_\infty \sum_{\lambda_i \in I} |\psi_i(x)\psi_i(y)| \le \sum_{\lambda_i \in I} |\psi_i(z_i)\psi_i(v_i)| \\
&\le \sum_{\lambda_i \in I} |\psi_i(v_i)| \, \mathrm{e}^{-mL} (C_d^2 L^d)^{-1} \sum_{(u,u')\in \partial \mathrm{B}_L(z_i)} |\psi_i(u)| \\
&\le \mathrm{e}^{-mL} \sum_{\lambda_i \in I} (C_d^2 L^d)^{-1} \sum_{(u,u')\in \mathbf{S}} |\psi_i(u)| \left(|\psi_i(x)| + |\psi_i(y)| \right) \\
&\le \mathrm{e}^{-mL} \frac{|\mathbf{S}|}{C_d^2 L^d} \max_{u \in \mathcal{G}} \sum_{\lambda_i \in I} \frac{1}{2} \left(|\psi_i(u)|^2 + |\psi_i(x)|^2 + |\psi_i(y)|^2 \right)
\end{aligned}$$

(using Bessel's inequality and $|\mathbf{S}| \le 2C_d^2 L^d$)

$$\le \mathrm{e}^{-mL} \, 2 \max_{u\in\mathcal{G}} \left(2 \| \, \mathbf{1}_u \, \|^2 + \| \, \mathbf{1}_x \, \|^2 + \| \, \mathbf{1}_y \, \|^2 \right) = 4\mathrm{e}^{-mL}.$$

Denote $\mathcal{S}_L = \{ \exists \, E \in I : \; \mathrm{B}_L(x) \text{ and } \mathrm{B}_L(y) \text{ are } (E,m)\text{-S} \}$, with $\mathbb{P}\{ \mathcal{S}_L \} \le \zeta(L)$, by assumption. Now we conclude:

$$\mathbb{E}\left[\mu_{x,y}(\phi) \right] = \mathbb{E}\left[\mathbf{1}_{\mathcal{S}_L} \, \mu_{x,y}(\phi) \right] + \mathbb{E}\left[\mathbf{1}_{\mathcal{S}_L^c} \, \mu_{x,y}(\phi) \right] \le \zeta(L) + 4\mathrm{e}^{-mL}. \quad \square$$

It is clear that the exponential term e^{-mL} can compete with $\zeta(L)$ only in applications to the FMM, for the MSA bounds on $\zeta(L)$ are at best sub-exponential in L. Otherwise, $\zeta(L)$ is the dominant term.

Note also that the decay rate of the bound $\zeta(L)$ sets natural restrictions on the class of the graphs $\mathcal{Z}$, due to the presence of the 'surface' factor $|\mathbf{S}|$, $\mathbf{S} = \mathbf{S}(L)$. In particular, only the FMM-based bounds have the chance to be efficient on trees and other graphs with exponentially growing balls.

8.2. **Dynamical localization on the entire graph.** Now one can make use of a simple argument employed earlier by Aizenman et al. [2, 3], in the framework of the FMM which always starts as a fixed-energy analysis. The quantities $\mu_{x,y}^{(H)}(\phi) = \langle \mathbf{1}_x \mid \phi(H) \mid \mathbf{1}_y \rangle$ defined, for example, for bounded continuous or Borel functions ϕ, generate signed (i.e., not necessarily positive) spectral measures associated with a self-adjoint operator H:

$$\int d\mu_{x,y}^{(H)}(E) \, \phi(E) := \langle \mathbf{1}_x \, | \phi(H) | \, \mathbf{1}_y \rangle.$$

In particular, we can consider, with $x, y, u \in \mathcal{Z}$ fixed, measures $\mu_{x,y}^k$ related to operators $H_{\mathrm{B}_{L_k}(u)}$, for all $k \ge 0$, as well as their counterparts $\mu_{x,y}$ for the operator H on the entire graph $\mathcal{Z}$. A sufficient condition for the vague convergence $\mu_{x,y}^k \to \mu_{x,y}$ as $k \to \infty$ is the strong resolvent convergence $H_{\mathrm{B}_{L_k}(u)} \to H$. Such convergence is well-known to occur for a very large class of operators, including (unbounded) Schrödinger operators in Euclidean spaces and their analogs on the so-called quantum graphs. Indeed, for (not necessarily bounded) operators H_n with a common core $\mathcal{D}$ to converge to an operator H with the same core, it suffices that $H_k\psi \to H\psi$ strongly for any element $\psi \in \mathcal{D}$ (cf. [22]). For finite-volume operators, one can usually find an appropriate core $\mathcal{D}$ formed by compactly supported functions ψ; for finite-difference Hamiltonians on graphs (even unbounded, e.g., for DSO with unbounded potentials) one can choose as $\mathcal{D}$ the subset of all functions with

finite supports. On such functions, $H_{\mathrm{B}_{L_k}(u)}\psi \to H\psi$ as $k\to\infty$ (by stabilization), therefore, the spectral measures converge vaguely: $\mu^k_{x,y}\to\mu_{x,y}$. By Fatou lemma, for any bounded Borel set $A\subset\mathbb{R}$, one has

$$|\mu_{x,y}(A)| \le \liminf_{k\to\infty} \left|\mu^k_{x,y}(A)\right|$$

(here $|\mu(A)| := \sup\{\mu(\phi),\ \|\phi\|\le 1,\ \mathrm{supp}\ \phi\subset A\}$). Taking the expectation and using the uniform upper bounds on EF correlators in finite balls, we conclude that

$$\mathbb{E}\left[\sup_{\phi\in\mathscr{B}_1} \left|\langle \mathbf{1}_x\,|\phi(H(\omega))|\,\mathbf{1}_y\rangle\right|\right] \le C\mathrm{e}^{-a\ln^{1+c}\mathrm{d}(x,y)} \tag{8.2}$$

(using the inequality $L_k^{-\kappa(1+\theta)^k}\le C\mathrm{e}^{-a\ln^{1+c}L_k}$, $C,a,c>0$). In particular, with functions $\phi_t:\lambda\mapsto \mathrm{e}^{-it\lambda}$, we obtain the strong dynamical localization property for the ensemble of random Hamiltonians $H(\omega)$.

9. Sub-exponential bounds on EF correlators without bootstrap

Now we will show how the polynomial (or slightly stronger than polynomial) decay bounds from Section 5 can be substantially improved and replaced by sub-exponential ones. Germinet and Klein proved in [20] a highly optimized and very general sub-exponential decay bound for a large class of random differential operators (an adaptation to lattices and graphs is straightforward). Unlike [20], our aim here is to obtain an elementary proof in the simplest situation, without a more involved bootstrap procedure. In the light of Sections 6–8, it suffices to work with the resolvents at a fixed energy $E\in I\subset\mathbb{R}$.

The advantage of the method presented below is that it gives rise to exponential bounds on the decay of eigenfunctions, while using an exponential sequence of scales, $L_{k+1}=YL_k$, as in [26] and in [20], gives directly only a sub-exponential bound. (Recall that [20] uses several multi-scale analyses to obtain final results, including *exponential* spectral localization. In [26], it was indicated that exponential localization requires scales $L_k\sim L_0^{\alpha^k}$.)

The main idea of the method described below is quite natural. The MSA induction shows clearly that the exponent $\kappa>0$ in the power-law bound of the form $\mathbb{P}\left\{\mathrm{B}_{L_k}(x) \text{ is } (E,m)\text{-S}\right\}\le L_k^{-\kappa}$ grows with the number K_k of allowed singular L_{k-1}-balls inside $\mathrm{B}_{L_k}(x)$. We allow for a growing number $K_k\sim L_k^c$, $c\in(0,1)$, and use an elementary probabilistic bound on such an event, close in spirit to the Poisson limit theorem. In [20], a similar effect is achieved by a refinement of an idea from [26]: replacing the sequence of scales $L_k\sim L_0^{\alpha^k}$, $\alpha>1$, by a slower growing sequence $L_k\sim Y^kL_0$, while keeping uniformly bounded the maximal number K_k of allowed singular cubes.

9.1. Multiple singular balls: a probabilistic estimate.

Lemma 9.1. *Suppose that for any ball $\mathrm{B}_{L_j}(x)\subset\mathrm{B}_{L_{j+1}}(u)$ one has*

$$\mathbb{P}\left\{\mathrm{B}_{L_j}(x)\ \textit{is}\ (E,m)\text{-S}\right\}\le \mathrm{e}^{-L_j^\delta},\ \delta>0.$$

Let $\mathcal{N}(\omega)$ be the maximal cardinality of collections of pairwise disjoint (E,m)-S balls $\mathcal{C}=\{\mathrm{B}_{L_j}(x_i), i=1,\dots,\mathcal{N}\}$. Then, for $\sigma>\delta$ and L_0 is large enough,

$$\mathbb{P}\left\{\mathcal{N}(\omega)\ge L_j^{\sigma(\alpha-1)}\right\}\le \frac{1}{2}\mathrm{e}^{-L_{j+1}^\delta}.$$

Proof. Fix a possible (unordered) configuration of centers x_i of disjoint (E,m)-S balls, $i=1,\dots,k$. Let $N=|\mathrm{B}_{L_{j+1}}(u)|$. The number of such configurations for a fixed k is bounded by $N(N-1)\cdots(N-k+1)/k! \le N^k/k!$, since choosing every center in the sequence $x_1,x_2,\dots$, excludes at least one possible position for the next center (indeed, many more). For a given configuration, the events $\{\mathrm{B}_{L_j}(x_i)$ is (E,m)-S$\}$ are independent, with probabilities $\le p:=\mathrm{e}^{-L_j^\delta}$, so

$$\mathbb{P}\{\mathcal{N}\ge n\}\le \sum_{k=n}^{N}\frac{N^k}{k!}p^k \le \frac{(Np)^n}{n!}\sum_{k=0}^{\infty}\frac{(Np)^k}{(k+n)!}\le \frac{(Np)^n}{n!}e^{Np}\le (Np)^n$$

for $p<N^{-1}$ and $n\ge 3$. With $N\le CL_j^{\alpha d}$ and $n:=[L_j^{\sigma(\alpha-1)}]$, one has $Np\le \mathrm{e}^{-L_j^\delta+C\ln L_j}$, thus

$$\begin{aligned}\mathbb{P}\{\mathcal{N}\ge n\}&\le (Np)^n\le \exp\left\{-\left(L_j^\delta-C\ln L_j\right)\left(L_j^{\sigma(\alpha-1)}-1\right)\right\}\\&\le \exp\left\{-\frac{1}{2}L_{j+1}^{\frac{\delta+\sigma(\alpha-1)}{\alpha}}\right\}.\end{aligned}$$

The condition $\frac{\delta+\sigma(\alpha-1)}{\alpha}>\delta$ is equivalent to the assumed inequality $\sigma>\delta$. Therefore, for some $\delta'>\delta$ and L_0 large enough

$$\mathbb{P}\left\{\mathcal{N}\ge L_j^{\sigma(\alpha-1)}\right\}\le \mathrm{e}^{-\frac{1}{2}L_{j+1}^{\delta'}}\le \frac{1}{2}\mathrm{e}^{-L_{j+1}^{\delta}}. \quad \square$$

9.2. **Decay of (ℓ,q)-subharmonic functions with "singular" points.** The radial descent bound given by Lemma 4.3 will require an adaptation.

Definition 9.1. Let $\mathcal{G}$ be a finite connected graph and $L\ge\ell\ge 0$ two integers and $q\in(0,1)$. Consider a ball $\mathrm{B}_L(u)\subsetneq\mathcal{G}$ and a function $f:\mathcal{G}\to\mathbb{R}_+$.

(1) We say that a point $x\in\mathrm{B}_L(u)$ is (ℓ,q)-regular for the function f iff
$$f(x)\le q\max_{y\in\mathrm{B}_{\ell+1}(x)}f(y)$$
and denote by $\mathcal{R}(f)(\subset\mathrm{B}_L(u))$ the set of all regular points of function f.

(2) Given a point $x\in\mathrm{B}_L(u)$, let $R(x)$ be the smallest integer such that $\mathrm{S}_{R(x)}\subset\mathcal{R}(f)$; if no such integer exists, we set formally $R(x)=+\infty$.

(3) We say that f is $(\ell,q,\mathcal{R})$-subharmonic in $\mathrm{B}_L(u)$, with $\mathcal{R}=\mathcal{R}(f)$, if for any point x with $R(x)<\infty$ and for all $r\ge 0$ such that
$$\mathrm{S}_{r+\ell+1}(u)\subset\mathrm{B}_{L+1}(u),\quad \mathrm{S}_r(u)\subset\mathcal{R},$$
one has
$$f(x)\le q\max_{y\in\mathrm{B}_{r+\ell+1}(x)}f(y). \tag{9.1}$$

Lemma 9.2. *Let a function $f:\mathcal{G}\to\mathbb{R}_+$, defined in a finite connected graph $\mathcal{G}$, be $(\ell,q,\mathcal{R})$-subharmonic in $\mathrm{B}_L(u)\subsetneq\mathcal{G}$. Suppose that the set $\mathcal{R}^c=\mathrm{B}_L(u)\setminus\mathcal{R}$ is covered by a family of annuli*

$$\mathbb{A}=\{A_i,1\le i\le n\},\ A_i=\mathrm{B}_{b_i}(u)\setminus\mathrm{B}_{a_i}(u),\ b_i-a_i\le c_i\ell, c_i\in\mathbb{N}^*,$$

of total width

$$w(\mathbb{A})=\sum_i(b_i-a_i)\le\sum_i c_i\ell=C\ell,$$

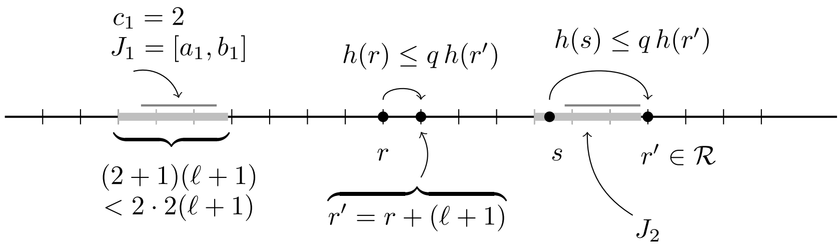

FIGURE 2. Example for the proof of Lemma 9.2. Recursion for $r \in \mathcal{R}$ (step of length $\ell + 1$) and for $s \in \mathcal{R}^c$ (step of length $\le |J_2|$).

with $C \in \mathbb{N}$ *and* $2C(\ell+1) < L$*.Then*

$$f(u) \le q^{\left\lfloor \frac{L+1}{\ell+1} \right\rfloor - 2C} \mathcal{M}(f, \mathcal{G}). \tag{9.2}$$

Proof. Divide the integer interval $[0, L+1] \cap \mathbb{N}$ into $N := \left\lfloor \frac{L+1}{\ell+1} \right\rfloor$ intervals of the form $I_j = [j(\ell+1), (j+1)(\ell+1)-1]$, $j = 0, 1, ..., n-1$, and, eventually, a remainder which will be unused in the argument. Call an interval I_j good if $\mathrm{S}_{j(\ell+1)} \subset \mathcal{R}$. Since $\mathrm{B}_L(u) \subsetneq \mathcal{G}$, the sphere $\mathrm{S}_{L+1}(u)$ is non-empty. The radial projection

$$\mathrm{B}_{L+1}(u) \ni x \mapsto \mathrm{d}(u, x) \in [0, L+1]$$

maps an annulus of width $c_i \ell$ onto an interval of length $c_i \ell$, covered by at most $c_i + 1 \le 2c_i$ adjacent intervals of the form I_j. Therefore, the entire set $\mathcal{R}^c$ is radially mapped onto a subset of $[0, L]$ covered by a family of at most $2C$ intervals I_j. Respectively, at least $K = N - 2C$ intervals $I_{j_1}, I_{j_2}, \dots, I_{j_K}$ must be good, and it follows from the hypotheses that $K \ge 1$. Further, let $h(r) := \max_{x \in \mathrm{B}_r(u)} f(x) \ge 0$, $r \in [0, L+1]$. Note that if $x \in \mathrm{B}_{j(\ell+1)}(u)$ and $z \in \mathrm{B}_{\ell+1}(x)$, then $z \in \mathrm{B}_{(j+1)\ell+1}(u)$. For $x \in \mathcal{R}(f)$ with $\mathrm{d}(u, x) =: r \in I_{j_i}$ one can apply (4.1), so for any $1 \le i \le K$, one has

$$\begin{aligned} h(j_i(\ell+1)) &= \max_{x \in \mathrm{B}_{j_i(\ell+1)}(u)} f(x) \le q \max_{z \in \mathrm{B}_{(j_i+1)(\ell+1)}(u)} f(z) \\ &= q h((j_i+1)(\ell+1)). \end{aligned} \tag{9.3}$$

The value $i = K$, i.e., the index j_K, is admissible here, since

$$(j_K + 1)(\ell+1) \le K(\ell+1) + 2C(\ell+1) \le L+1,$$

and the set $\mathrm{S}_{L+1}(u)$ is nonempty, by assumption. For $i = K$, we have, by monotonicity of the function h, the following upper bound on the RHS of (9.3):

$$q h((j_i+1)(\ell+1)) \le q h(L+1) \le q \|f\|_\infty,$$

so by backward recursion $i = K-1, ..., 1$, one obtains in $K-1$ steps:

$$f(u) = h(0) \le h(j_1(\ell+1)) \le q^K \|f\|_\infty \le q^{\left\lfloor \frac{L+1}{\ell+1} \right\rfloor - 2C} \|f\|_\infty. \quad \square$$

Definition 9.1 is tailored so as to suit the following analog of Lemma 4.2.

Lemma 9.3. *Consider a finite graph $\mathcal{G}$, a DSO $H_{\mathcal{G}}$ and a ball $\mathrm{B}_{L_{k+1}}(u) \subsetneq \mathcal{G}$. Fix some $E \in \mathbb{R}$, assume that $\mathrm{B}_{L_{k+1}}(u)$ is E-CNR, and let $\mathcal{R} = \mathcal{R}(\mathcal{G}, E)$ be the set of points $x \in \mathcal{G}$ such that the ball $\mathrm{B}_{L_k}(x) \subset \mathcal{G}$ is (E, m)-NS. Then for any $y \in \mathcal{G} \setminus \mathrm{B}_{L_{k+1}}(u)$, the function $x \mapsto \left|G_{\mathrm{B}_{L_{k+1}}(u)}(x, y; E)\right|$ is $(L_k, q, \mathcal{R})$-subharmonic in $\mathrm{B}_{L_{k+1}}(u)$, with $q = e^{m(1+\frac{1}{2}L_k^{-\tau})L_k}$, if L_0 is large enough.*

Proof. If $x \in \mathcal{R}$, then, by the (E, m)-NS property of $\mathrm{B}_{L_k}(x)$,

$$f(x) \le \mathrm{e}^{-\gamma(m,L_k)L_k} \mathcal{M}(f, \mathrm{B}_{L_k+1}(x)).$$

For $x \in \mathcal{R}^{\mathrm{c}}$ with $R(x) < \infty$ (the radius $R(x)$ is defined as in Definition 9.1), we have, by E-NR property of the ball $\mathrm{B}_{R(x)-1}(u)$, stemming from the assumed E-CNR property of $\mathrm{B}_{L_{k+1}}(u)$,

$$\begin{aligned} f(x) &\le |\partial \mathrm{B}_{L_k}(x)| \, \|G_{\mathrm{B}_{R(x)-1}(u)}(x, v; E)\| \max_{\mathrm{d}(u,v')=R(x)} |G_{\mathrm{B}_{L_{k+1}}(u)}(v', y; E)| \\ &\le \mathrm{e}^{L_k^\beta} \max_{\mathrm{d}(u,v')=R(x)} |G_{\mathrm{B}_{L_{k+1}}(u)}(v', y; E)| \end{aligned}$$

and since $S_{R(x)}(u) \subset \mathcal{R}$,

$$\begin{aligned} &< \mathrm{e}^{L_k^\beta} \max_{v' \in S_{R(x)}(u)} \mathrm{e}^{-\gamma(m,L_k)L_k} \mathcal{M}(f, \mathrm{B}_{1+L_k}(v')) \\ &\le \mathrm{e}^{-\gamma(m,L_k)L_k + L_k^\beta} \mathcal{M}(f, \mathrm{B}_{R(x)+L_k+1}(u)). \end{aligned}$$

For L_0 large enough and $\tau < 1-\beta$, one has $\gamma(m, L_k)L_k - L_k^\beta \ge m(1+\frac{1}{2}L_k^{-\tau})L_k$. $\square$

9.3. **Scaling with sub-exponential bounds.** From this point on, we fix the key parameters used in the scale induction as follows:

$\alpha = \frac{4}{3}$	$\beta = \frac{1}{3}$
$\delta = \frac{1}{4} < \beta$	$\sigma = \frac{1}{3} > \delta$
$\rho = \frac{1}{3}$	$\tau = \frac{1}{8} = \frac{1}{2}\big(\rho - \sigma(\alpha - 1)\big)$

Definition 9.2. A ball $\mathrm{B}_{L_{k+1}}(x)$ is called E-tunneling (E-T) if, for some $E \in I$, it contains a collection of more than $L_k^{\sigma(\alpha-1)}$ pairwise disjoint (E, m)-S balls of radius L_k, and E-non-tunneling (E-NT), otherwise.

Lemma 9.4. *If a ball $\mathrm{B}_{L_{k+1}}(u)$ is E-NT and E-CNR, then it is (E, m)-NS.*

Proof. Fix two points $x, y \in \mathrm{B}_{L_{k+1}}(u)$ with $\mathrm{d}(x, y) \ge L_k^{1+\varrho}$ and let $R = \mathrm{d}(x, y) - L_k$. By Lemma 9.3, the function $f : z \mapsto G_{\mathrm{B}_{L_{k+1}}}(x, z)$ is $(L_k, q, \mathcal{S})$-subharmonic in $\mathrm{B} = \mathrm{B}_R(x)$, with $q = \mathrm{e}^{-m(1+\frac{1}{2}L_k^{-\tau})L_k}$, and the E-NT assumption guarantees that $\mathcal{S}$ can be covered by at most $L_k^{\sigma(\alpha-1)}$ balls of radius $2L_k$, hence by a collection $\mathbb{A}$ of annuli centered at u of total width $w(\mathbb{A}) \le L_k^{\sigma(\alpha-1)} \cdot 4L_k$. Therefore,

$$f(x) \le q^{\frac{R - w(\mathbb{A}) - 2L_k}{L_k+1}} \|f\|_\infty \le q^{\frac{R - 5L_k^{1+\sigma(\alpha-1)}}{L_k+1}} \|f\|_\infty. \tag{9.4}$$

Recall that $\varrho - \sigma(\alpha - 1) = 2\tau$, so

$$R - 5L_k^{1+\sigma(\alpha-1)} > R\left(1 - 5L_k^{\sigma(\alpha-1)-\varrho}\right) = R\big(1 - 5L_k^{-2\tau}\big).$$

Put this lower bound into (9.4) and write, with the convention $-\ln 0 = +\infty$:

$$\begin{aligned} -\ln f(x) &\ge -\ln \left\{ \left(e^{-m(1+\frac{1}{2}L_k^{-\tau})L_k} \right)^{\frac{R\left(1-5L_k^{-2\tau}\right)}{L_k+1}} e^{L_{k+1}^{\beta}} \right\} \\ &\ge mR \left(\left(1+\tfrac{1}{2}L_k^{-\tau}\right) \tfrac{L_k}{L_k+1} \left(1-5L_k^{-2\tau}\right) - \frac{L_{k+1}^{\beta}}{mR} \right) \end{aligned}$$

(with $m \ge 1$, $R = \mathrm{d}(x,y) - L_k > L_k^{1+\varrho} - L_k > \frac{1}{2}L_k^{1+\varrho}$ and L_0 large enough)

$$\ge mR \left((1+\tfrac{1}{2}L_k^{-\tau}) \left(1-6L_k^{-2\tau}\right) - 2L_k^{-1-\varrho+\alpha\beta} \right)$$

(use $\rho = \frac{1}{3}$, $\alpha\beta = \frac{4}{9} \Rightarrow 1+\rho-\alpha\beta = \frac{8}{9} > 2\tau$)

$$\begin{aligned} &\ge mR\left(1+\tfrac{1}{2}L_k^{-\tau}\right)\left(1-7L_k^{-2\tau}\right) \ge m\left(1+\tfrac{1}{4}L_k^{-\tau}\right) \mathrm{d}(x,y) \\ &\ge \gamma(m, L_{k+1})\, \mathrm{d}(x,y). \end{aligned}$$

□

Consider the following property which we will prove by induction for all $k \in \mathbb{N}$:

$\mathsf{S}(k,E)$: For any ball $\mathrm{B}_{L_k}(x) \subset \mathcal{Z}$, one has

$$\mathbb{P}\left\{ \mathrm{B}_{L_k}(x) \text{ is } (E,m)\text{-S} \right\} \le \mathrm{e}^{-L_k^{\delta}}. \tag{9.5}$$

Theorem 9.5. $\mathsf{S}(k,E)$ *implies* $\mathsf{S}(k+1,E)$.

Proof. Denote by $N(\mathrm{B}_{L_{k+1}}(x))$ the maximal cardinality of collections of pairwise disjoint (E,m)-S balls of radius L_k inside $\mathrm{B}_{L_{k+1}}(x)$. Introduce the events

$$\begin{aligned} \mathcal{B}_{k+1} &= \left\{ N(\mathrm{B}_{L_{k+1}}(x)) \ge L_k^{\sigma(\alpha-1)} \right\}, \\ \mathcal{E}_{k+1} &= \left\{ \mathrm{B}_{L_{k+1}}(x) \text{ is } E\text{-PR} \right\}. \end{aligned}$$

By Lemma 9.4, $\{\omega :\ \mathrm{B}_{L_{k+1}}(x) \text{ is } (E,m)\text{-S}\} \subset \mathcal{E}_{k+1} \cup \mathcal{B}_{k+1}$. By Wegner estimate (2.8), using $\beta > \delta$, we have

$$\mathbb{P}\left\{ \mathcal{E}_{k+1} \right\} \le \tfrac{1}{2}\mathrm{e}^{-L_k^{\beta}} \le \tfrac{1}{2}\mathrm{e}^{-L_k^{\delta}}$$

so the claim follows from Lemma 9.1 saying that $\mathbb{P}\left\{ \mathcal{B}_{k+1} \right\} \le \frac{1}{2}\mathrm{e}^{-L_k^{\delta}}$. □

For the initial scale bound $\mathsf{S}(0)$ (indeed, *any* desired probabilistic bound, cf. [12]) to hold true, it suffices to pick $|g|$ large enough. Therefore, we come by induction to the following

Theorem 9.6. *Assume that the random potential fulfills the regularity condition*[2] *(2.7). Then for all $|g|$ large enough, the property $\mathsf{S}(k)$ holds true for all $k \ge 0$.*

Remark 9.1. A more tedious (but elementary) parametric analysis shows that one can actually choose the exponent $\delta \in (0,1)$ arbitrarily close to 1, thus getting a

[2]As was said, the Lipshitz continuity condition (2.7) can be relaxed to the Hölder continuity.

sub-exponential decay very close to the exponential one. Indeed, the complete set of requirements for the scaling parameters is as follows:

$$
\begin{aligned}
&0 < \rho < \alpha - 1 \\
&0 < \sigma < \frac{\rho}{\alpha - 1}, \quad 0 < \delta < \min\{\beta, \sigma\} \\
&0 < 2\tau = \min\{\rho - \sigma(\alpha - 1), 1 + \rho - \alpha\beta\}
\end{aligned}
$$

A direct inspection shows that for any $\epsilon \in (0, \frac{1}{2})$, one can set, e.g.,

$\alpha = 1 + 4\epsilon$	$\beta = 1 - \epsilon$
$\delta = 1 - 2\epsilon < \beta$	$\sigma = 1 - \epsilon > \delta$
$\rho = 4\epsilon - 2\epsilon^2$	$\tau = \epsilon^2 = \frac{1}{2}\big(\rho - \sigma(\alpha - 1)\big)$

10. On a direct approach to the decay of eigenfunctions

We have seen that the fixed-energy analysis of resolvents implies the spectral and dynamical localization. On the other hand, it is also possible to adapt the approach from the Spencer's work [26] to a direct analysis of eigenfunctions in arbitrarily large finite balls. For single-particle models, such an adaptation has been proposed in our recent paper [8]. The new scaling scheme can be summarized as follows:

- The key notion becomes that of an m-localized ball. We say that a ball $\mathrm{B}_L(u)$ is m-localized if the eigenbasis $\{\psi_i\}$ of the operator $H_{\mathrm{B}_L(u)}$ fulfills the following condition: for all points $x, y \in \mathrm{B}_L(u)$ with $\mathrm{d}(x, y) \geq L^{7/8}$,
$$\sum_{\lambda_i \in \Sigma(H_{\mathrm{B}_L(u)})} |\psi_i(x)\, \psi_i(y)| \leq \mathrm{e}^{-\gamma(m,L)\mathrm{d}(x,y)}.$$
- A ball $\mathrm{B}_{L_{k+1}}(u)$ is called E-completely non-resonant (E-CNR) if it is E-NR and contains no E-R ball of radius $\geq L_k$.
- It is readily seen from the eigenfunction expansion of the resolvent that an m-localized ball which is E-NR must be (E, m)-NS.
- A direct analog of Lemma 5.1 is still valid.
- Consider the bounds (which we will denote by $\mathsf{LOC}(k)$, $k \geq 1$) of the form
$$\mathbb{P}\left\{ \mathrm{B}_{L_k}(u) \text{ is } m\text{-localized } \right\} \geq 1 - L_k^{-\kappa(1+\theta)^k}. \tag{10.1}$$
The initial scale bound $\mathsf{LOC}(1)$, for any $m \geq 1$, $L_0 \geq 1$ and $|g|$ large enough, is easily obtained by elementary perturbation theory for self-adjoint operators (which are finite-dimensional here) with simple spectrum. The induction step ($\mathsf{LOC}(k) \Rightarrow \mathsf{LOC}(k+1)$) is performed as follows.
  - Assume $\mathsf{LOC}(k)$ and consider a ball $\mathrm{B}_{L_{k+1}}(u)$. If it is not m-localized, then by an analog of Lemma 5.1, for some $E \in \mathbb{R}$ it must contain two disjoint (E, m)-S balls $\mathrm{B}_{L_k}(x)$, $\mathrm{B}_{L_k}(y)$. One can easily infer from the Wegner estimate that with probability $\geq 1 - \mathrm{e}^{-L_{k+1}^{\beta/2}}$, for any $E \in \mathbb{R}$ either $\mathrm{B}_{L_k}(x)$ or $\mathrm{B}_{L_k}(y)$ is E-CNR.
  - If $\mathrm{B}_{L_k}(x)$ is (E, m)-S and E-CNR, it must contain a pair of disjoint m-non-localized balls of radius L_{k-1}. By virtue of the inductive assumption $\mathsf{LOC}(k)$, the probability of the latter event is bounded by $CL_{k-1}^{2d\alpha^2 - 2\kappa(1+\theta)^{k-1}}$.

– For $\alpha \in (1, \sqrt{2})$, $\kappa > \frac{2\alpha^2 d}{2-\alpha^2}$ and an appropriately chosen $\theta > 0$, the above bounds imply $\mathsf{LOC}(k+1)$.

The property $\mathsf{LOC}(k)$, proven for all $k \geq 0$, is already a form of localization of eigenfunctions. In addition, it implies the usual variable-energy MSA bounds, hence, the strong dynamical localization and an exponential decay of eigenfunctions.

It is to be emphasized that the main analytic tool of the simplified MSA remains the elementary Lemma 4.1 (and Lemma 4.3 easily stemming from it). As was mentioned earlier, the idea of the "two-sided" estimates of the Green functions (and, similarly, eigenfunctions) goes back to Spencer's work [26].

Conclusion

We have shown that the fixed-energy probabilistic analysis of random Anderson-type Hamiltonians implies, in a fairly general and elementary way, stronger manifestations of the Anderson localization phenomenon, viz.: spectral and strong dynamical localization. The new method, going in the same direction as the well-known Simon–Wolf criterion of localization, applies also to multi-particle systems (as will be shown in our forthcoming work) for which no analog of the Simon–Wolf approach has been developed so far. Moreover, combined with the simplified Germinet–Klein argument, it gives rise to the strong dynamical localization, not only spectral localization. Therefore, a very simple scaling procedure going back to Spencer's work [26], as well as its counterpart for interacting multi-particle systems, results in a simple proof of strong dynamical localization for a large class of models.

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